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Mayer Expansions and the Hamilton–Jacobi Equation

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We review the derivation of Wilson's differential equation in (infinitely) many variables, which describes the infinitesimal change in an effective potential of a statistical mechanical model or quantum field theory when an infinitesimal "integration out" is performed. We show that this equation can be solved for short times by a very elementary method when the initial data are bounded and analytic. The resulting series solutions are generalizations of the Mayer expansion in statistical mechanics. The differential equation approach gives a remarkable identity for "connected parts" and precise estimates which include criteria for convergence of iterated Mayer expansions. Applications include the Yukawa gas in two dimensions past the $\beta = 4\pi$ threshold and another derivation of some earlier results of Göpfert and Mack.

KEY WORDS: Multiscale Mayer expansions; renormalization group; tree graph identities.

1. INTRODUCTION

Many problems in statistical mechanics and quantum field theory center on the analysis of functional integrals of the form

$$Z(\varphi') = \int d\mu(\varphi) \exp[-V(\varphi + \varphi')] \equiv [\mu * \exp(-V)](\varphi') \qquad (1.1)$$

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where $\boldsymbol{\varphi} = (\varphi_x)_{x \in \Lambda}$ is a Gaussian process with joint probability distribution $d\mu$ with mean zero and covariance C_{xy}, so that

$$\int d\mu(\boldsymbol{\varphi})\, \varphi_x = 0 \qquad \text{all } x \in \Lambda$$

$$\int d\mu(\boldsymbol{\varphi})\, \varphi_x \varphi_y = C_{xy} \qquad \text{all } x, y \in \Lambda \tag{1.2}$$

and V is an approximately *additive* functional (*local* in physics terminology). This means V has the form

$$V \simeq \sum_{x \in \Lambda} v_x(\varphi_x) \tag{1.3}$$

where the precise meaning of $\simeq$ will be discussed later.

The object in analyzing Z is to determine information on its dependence on parameters in V in the limit as the index set Λ increases to an infinite set. In applications there is always a natural procedure available to define the covariance matrix and a sequence of V's for the sets through which Λ increases to an infinite set.

An easy version of this problem occurs when V is exactly additive and the covariance matrix vanishes off the diagonal, because in this case Z factors into a product of one-dimensional integrals. The expansion techniques of statistical mechanics quantify what happens near this case. In recent years the renormalization group philosophy has made it clear that the best results are obtained when one applies these expansions to small "subintegrals" of $\int d\mu$ rather than the whole integral at once. For example, one can write the covariance C as the sum of two (or more) covariances $C = C^{(1)} + C^{(2)}$ with corresponding Gaussian processes $\boldsymbol{\varphi}^{(1)}$, $d\mu^{(1)}$ and $\boldsymbol{\varphi}^{(2)}$, $d\mu^{(2)}$ so that

$$Z = \mu^{(1)} * \mu^{(2)} * \exp(-V) = \mu^{(1)} * \exp\{\log[\mu^{(2)} * \exp(-V)]\}$$

Since the original covariance can be written as a sum of many covariances $C^{(1)},..., C^{(N)}$ the problem becomes a study of the map

$$V \to -\log(\mu^{(i)} * e^{-V}) \tag{1.4}$$

This or some variant map has been used and studied in very many papers; see, for example, Ref. 1. The present paper is concerned with the limiting case of this philosophy, where we write

$$\boldsymbol{\varphi} = \int d\boldsymbol{\varphi}(t), \qquad C = \int dt\, \dot{C}(t) \tag{1.5}$$

Each increment $d\varphi(t)$ is an independent Gaussian process with an infinitesimal covariance $\dot{C}(t)\,dt$ (i.e., an Ito integral increment). This leads to a differential equations approach: a flow equation describing how V must change to compensate for an infinitesimal change in C. Such equations were first obtained by Wilson,[2] but have not yet been much used in mathematical investigations. Polchinski[3] is an exception, which started our interest in this approach.

The flow equation obtained in this way is a partial differential equation of the form

$$\partial V/\partial t = \tfrac{1}{2}(V_{\varphi\varphi} - V_{\varphi}^2) \tag{1.6}$$

where the $\varphi\varphi$ subscript denotes a Laplacian in all the φ's. Since there are an infinite number of φ variables in the limit as Λ grows to an infinite set, even if V is local, it is an infinite sum of roughly equal functions and will typically be infinite. On the other hand, its derivatives with respect to the variables φ_x can remain finite. In this paper we study these derivatives of solutions that are obtained by "the method of variation of parameters." These are power series in a parameter in front of the nonlinear term and they turn out to be generalizations of the Mayer expansion in statistical mechanics. The study of their analogues in the discrete case was begun by Gallavotti *et al.* and is reviewed in Ref. 4. Related ideas were introduced by Göpfert and Mack.[12]

The maximum principle, i.e., positivity of the fundamental solution of the heat equation, along with a Cauchy–Kowaleska type of existence proof shows that the derivatives of V may be estimated by derivatives of the solution of a similar equation with the Laplacian omitted and a sign changed, i.e.,

$$\partial V/\partial t = \tfrac{1}{2} V_{\varphi} \cdot V_{\varphi} \tag{1.7}$$

Furthermore, we can even estimate suitable norms of the original V (which measure how "nonlocal" V is) in terms of the solution of (1.8) with only one φ variable. A Burger's equation was analyzed in Ref. 5 in a similar way to obtain mean-field theory bounds on the Ising model. Equation (1.7) is a Hamilton–Jacobi equation with an unusual sign (or alternatively the initial data have been flipped in sign.). The Hamilton–Jacobi equation is solved by the action principle of classical mechanics (see, for example, Ref. 19). Furthermore, in perturbation theory the same classical action is given by summing over all tree graphs. Thus the domination of the flow equation by a Hamilton–Jacobi equation provides a new viewpoint on the tree graph bounds used in cluster expansions. See the review article in Ref. 7 for discussion and references on this topic. Unfortunately, the "wrong sign" in

Eq. (1.7) means that we have lost relative signs between tree graphs and it is in these signs that the stability of the interaction is encoded. For this reason we are forced to restrict ourselves to initial data with *bounded derivatives*. This is a bad restriction as far as interesting physical systems are concerned and it would be most desirable to get bounds involving a Hamilton–Jacobi equation with the standard sign. In particular, it is important to note that our theorems are too weak to use as a basis for a good global existence theorem such as is needed for the study of critical behavior by the renormalization group, in contrast to the methods discussed by several other authors.[1] Even for systems with initial data in our class we do not obtain long-time existence theorems useful in the study of critical phenomena, because under scaling, which is part of the renormalization group, bounded functions become less bounded and start to resemble polynomials.

Nevertheless, this method is very simple and useful as far as it goes and provides accurate estimates. We show in Extension 2.5 in Section 2 and in Section 4 that when the initial data are chosen to be a trigonometric polynomial the existence theorem can be improved in a simple way so as to take advantage of the smoothing properties of the Laplacian in the flow equation and then the method is strong enough to prove convergence of the Mayer expansion ($\equiv$ perturbation theory) for the continuum sine-Gordon field theory for $\beta < 16\pi/3$ and z small. The sine-Gordon field theory is the same as the two-dimensional Yukawa gas and $16\pi/3$ is beyond the first threshold at $\beta = 4\pi$ at which the gas collapses into dipoles. These are mostly not new results, but we think there is a conceptual advance in our derivation. We have also found a very useful graphical identity (Theorem 3.1), which is representative of the principle that graphical expansions are dominated by their tree graphs. As an application of this result we apply the identity to the (Villain) Yukawa system in three dimensions, first analyzed by Göpfert and Mack.[12]

Each of the following sections refers to results in earlier sections, but nevertheless can be read almost independently.

2. FLOW EQUATIONS AND A SHORT-TIME EXISTENCE THEOREM

In this section we study the approximate additivity properties of $V(t, \varphi)$ defined by

$$V := -\log[\mu_t * \exp(-V^{(0)})] \tag{2.1}$$

where $d\mu_t$ is the mean zero Gaussian joint probability distribution with covariance $C(t) = C_{xy}(t)$ where $t \in \mathbf{R}^+$ is a real number that parametrizes a differentiable deformation of the covariance. Thus, we assume

$$C(0) = 0 \tag{2.2a}$$

$$C \text{ is differentiable} \tag{2.2b}$$

$$\dot{C} \equiv dC/dt \geqslant 0 \tag{2.2c}$$

and interpret $d\mu_0$ to be the measure $\delta(\boldsymbol{\varphi})$. This is consistent with continuity because the weak limit of $d\mu_t$ as t tends to zero is $\delta(\boldsymbol{\varphi})$. Condition (2.2c) means positivity in the sense of forms. $V^{(0)}$ is a smooth, bounded function of $\boldsymbol{\varphi}$. We will also use $\mu_{[t,s]}$ to denote the Gaussian measure with covariance $\int_s^t \dot{C}(\tau)\, d\tau$, so, for example, μ_t is the same as $\mu_{[t,0]}$.

Lemma 2.1. V is the unique bounded solution to

$$\frac{\partial V}{\partial t} = \frac{1}{2} \sum_{x,y} \dot{C}_{xy}(t) \left(\frac{\partial^2 V}{\partial \varphi_x\, \partial \varphi_y} - \frac{\partial V}{\partial \varphi_x} \frac{\partial V}{\partial \varphi_y} \right); \qquad t > 0$$

$$\lim_{t \to 0} V(t) = V^{(0)} \tag{2.3}$$

Proof. μ_t has a Gaussian density which is the fundamental solution for a heat equation, so that $Z \equiv \mu_t * Z^{(0)}$ is the unique bounded solution to

$$\frac{\partial Z}{\partial t} = \frac{1}{2} \sum_{x,y} \dot{C}_{xy} \frac{\partial^2 Z}{\partial \varphi_x\, \partial \varphi_y}; \qquad \lim_{t \to 0} Z(t, \varphi) = Z^{(0)} \tag{2.4}$$

The proof is now an elementary calculation using $V = -\log(\mu_t * Z^{(0)})$ with $Z^{(0)} \equiv \exp(-V^{(0)})$. ∎

To state our existence theorem we define

$$V_{x_1,\ldots,x_M} \equiv \frac{\partial^M V}{\partial \varphi_{x_1} \cdots \partial \varphi_{x_m}} \tag{2.5}$$

$$V_M(t) \equiv \sup_{\boldsymbol{\varphi} \text{ real},\, x} \frac{1}{M!} \sum_{x_2,\ldots,x_M} |V_{x,x_2,\ldots,x_M}(t, \boldsymbol{\varphi})| \tag{2.6}$$

$$|\dot{C}(t)| \equiv \sup_x \sum_y |\dot{C}_{xy}(t)|$$

V_M is defined for $M = 1$ by omitting the sum in (2.6).

Theorem 2.2. Suppose that the power series *in one variable* φ

$$v^{(0)}(\varphi) \equiv \sum_{M=1}^{\infty} V_M(0)\, \varphi^M \tag{2.7}$$

has a nonzero radius of convergence; then

$$\frac{\partial v}{\partial t}=\frac{1}{2}|\dot{C}(t)|\left(\frac{\partial v}{\partial \varphi}\right)^2, \qquad v(0,\varphi)=v^{(0)}(\varphi) \tag{2.8}$$

defines for t small a function $v(t,\varphi)$ that is analytic near $\varphi=0$. For all t for which v exists the flow equation (2.3) has a unique solution which is analytic in the initial data and bounded according to

$$V_M(t)\leqslant v_M(t), \qquad M=1,2,... \tag{2.9}$$

where $v_M(t)$ is the Mth coefficient of the power series for v:

$$v(t,\varphi)=\sum_{M=0}^{\infty} v_M(t)\,\varphi^M \tag{2.10}$$

In particular these bounds hold if t is sufficiently small that

$$\left[\int_0^t |\dot{C}(s)|\,ds\right]\sup_{M\geqslant 1}\,[MV_M(0)]^{2/M}<\frac{1}{4} \tag{2.11}$$

Proof. The flow equation is

$$\frac{\partial V}{\partial t}=\frac{1}{2}\sum_{y,z}\dot{C}_{yz}(t)\left(\frac{\partial^2 V}{\partial\varphi_y\,\partial\varphi_z}-\frac{\partial V}{\partial\varphi_y}\frac{\partial V}{\partial\varphi_z}\right), \qquad x\in\Lambda \tag{2.12}$$

Since we are only considering bounded solutions of these equations, we can rewrite them as integral equations

$$V(t)=\mu_t * V^{(0)}-\frac{1}{2}\int_0^t ds\sum_{y,z}\dot{C}_{yz}(s)\,\mu_{[t,s]} * [V_y(s)\,V_z(s)]$$

We differentiate this with respect to $\varphi_{x_1},..., \varphi_{x_M}$ and obtain

$$V_I=\mu_t * V_I^{(0)}-\frac{1}{2}\sum_{J\subset I}\int_0^t ds\sum_{y,z}\dot{C}_{yz}(s)\,\mu_{[t,s]} * (V_{y,J}V_{z,I\setminus J}) \tag{2.13}$$

where $I\equiv\{1,..., M\}$ and we have used the set subscripts I, J, and $I\setminus J$ to denote derivatives with respect to φ's at x_i for $i\in J$, etc. We can insert these equations into themselves and thereby generate a series for any given derivative with coefficients involving only derivatives of the initial conditions. This becomes a power series in z if we replace $V^{(0)}$ by $zV^{(0)}$. We also find by taking absolute values and supremums over φ that

$$V_M(t)\leqslant V_M(0)+\frac{1}{2}\sum_{p=0}^{M}\int_0^t ds\,|\dot{C}(s)|\,(p+1)\,V_{p+1}(s)$$
$$\times(M-p+1)\,V_{M-p+1}(s)$$

for $M \geqslant 1$. The μ's disappear on taking supremums because they are probability measures. The iteration of

$$v_M(t) = v_M(0) + \frac{1}{2} \sum_{p=0}^{M} \int_0^t ds \, |\dot{C}(s)| \, (p+1) \, v_{p+1}(s) \times (M-p+1) \, v_{M-p+1}(s) \tag{2.14}$$

where $M \geqslant 0$, produces a series that majorizes the formal series for V_M (with $M \geqslant 1$) obtained by iterating (2.13). Furthermore, if we define $v(t, \varphi)$ by

$$v(t, \varphi) \equiv \sum_M v_M(t) \, \varphi^M$$

then (2.14) becomes

$$v(t, \varphi) = v(0, \varphi) + \frac{1}{2} \int_0^t ds \, |\dot{C}(s)| \left(\frac{\partial v(s, \varphi)}{\partial \varphi} \right)^2 \tag{2.15}$$

which is the integral equation corresponding to

$$\frac{\partial v}{\partial t} = \frac{1}{2} |\dot{C}| \left(\frac{\partial v}{\partial \varphi} \right)^2, \qquad v(t=0) = \sum_{M=1}^{\infty} v_M(0) \, \varphi^M \tag{2.16}$$

We will now prove that the iteration of (2.14) converges, by replacing $v(0)$ by $zv(0)$ in (2.15) and showing that this equation has a solution analytic in z in a neighborhood of $\{z: |z| \leqslant 1\}$ and in φ near $\varphi = 0$ for t satisfying (2.11). It is enough to replace the initial condition on v by the majorizing series

$$v(0, \varphi) = z \sum_M \frac{1}{M} K^M \varphi^M = -z \log(1 - K\varphi)$$

where $K \equiv \sup_{M \geqslant 1} [M V_M(0)]^{1/M}$. By changes of variable

$$\tau = K^2 \int_0^t |\dot{C}(s)| \, ds, \qquad \varphi = K\varphi$$

the equations are transformed to

$$\frac{\partial v}{\partial \tau} = \frac{1}{2} \left(\frac{\partial v}{\partial \varphi} \right)^2, \qquad v(\tau = 0) = -z \log(1 - \varphi)$$

This equation may be solved by the action principle (see, for example, Ref. 19). Its solution is

$$v = -z \log(1 - \psi_0) - \frac{1}{2\tau} (\psi - \psi_0)^2$$

where ψ_0 is such that the right-hand side is critical, i.e.,

$$\frac{z}{1-\psi_0}+\frac{1}{\tau}(\psi-\psi_0)=0$$

v is analytic near $\varphi=0$ for $0\leqslant|z\tau|<1/4$, which by our change of variable above is the same as (2.11) since $|z|\leqslant 1$. We conclude from these arguments that the series generated by iterating (2.13) and its derivatives is convergent for small time in the V_M norms defined above. This implies that the series are uniformly convergent series of analytic functions, so that the series for V solves the flow equations (2.12). ∎

Some Extensions

Extension 2.3. If the set Λ has a metric ρ (example: Euclidean distance on $\Lambda\equiv\mathbf{Z}^d$), then we can discuss exponential localization, i.e., exponential decay of correlation functions, by defining

$$\rho(x_1,\ldots,x_N)=\inf_{T\in\text{tree graphs on }\{1,\ldots,N\}}\sum_{l,m\in T}\rho(x_l,x_m) \tag{2.17}$$

which is a measure of how spread out the index points $x_1,\ldots,x_N$ are. Next we fix $\lambda\geqslant 0$ and define

$$\begin{aligned}V_{M,\lambda}(t)&\equiv\sup_{\boldsymbol{\varphi}\text{ real},\,x}\frac{1}{M!}\sum_{x_2,\ldots,x_M}|V_{x,x_2,\ldots,x_M}(t,\boldsymbol{\varphi})|\exp[\lambda\rho(x,x_2,\ldots,x_M)]\\|\dot{C}(t)|_\lambda&\equiv\sup_x\sum_y|\dot{C}_{xy}(t)|\exp[\lambda\rho(x,y)]\end{aligned} \tag{2.6'}$$

With the same proof as in Theorem 2.2 we obtain the following result.

Theorem 2.2′. The same as Theorem 2.2 with V_M and $|\dot{C}|$ replaced by $V_{M,\lambda}$ and $|\dot{C}|_\lambda$, respectively.

Extension 2.4. If the index set Λ is a subset of the lattice $(\varepsilon\mathbf{Z})^d\subset\mathbf{R}^d$, then by replacing $V^{(0)}$ by $\varepsilon^d V^{(0)}$ all we have done carries over with sums over indices $x\in\Lambda$ replaced by Riemann sums. This means *there will be a continuum version of these results* in which derivatives with respect to φ_x are replaced by variational derivatives $\partial/\partial\varphi(x)$, $\boldsymbol{\varphi}$ becomes a Gaussian field over $\mathbf{R}^d$, etc. In particular, the flow equations can be reformulated as integral equations.

Extension 2.5. Theorem 2.2 does not take advantage of the smoothing properties of convolution by μ. As we will explain below, trigonometric polynomials are a standard choice for the initial data. In this special case it is easy to do better: fix $T \geqslant 0$ and consider, for example, the initial data

$$V^{(0)} = z \sum_x :\exp(i\varphi_x):_T$$

where $:\ :_T$ (normal ordering with respect to the covariance at $t = T$) is defined by

$$:\exp(i\varphi_x):_T \equiv \exp[\tfrac{1}{2} C_{xx}(T)] \exp(i\varphi_x)$$

We assume that $\dot{C}_{xx}$ is independent of x and set $c(s) \equiv \dot{C}_{xx}(s)$.

Proposition 2.6. For T sufficiently small that

$$z \int_0^T |\dot{C}(s)| \exp\left[\int_s^T c(\tau)\, d\tau\right] \mathrm{ds} < e^{-1}$$

The equation

$$\frac{\partial \bar{v}}{\partial t} = \frac{1}{2} \exp\left[\int_t^T c(\tau)\, d\tau\right] |\dot{C}(t)| \left(\frac{\partial \bar{v}}{\partial \varphi}\right)^2$$

$$\bar{v}(t = 0) = z e^{\varphi}$$

has a solution at least up to time T and for $M \geqslant 1$,

$$V_M(T) \leqslant v_M(T)$$

If Theorem 2.2 were applied, we would achieve essentially the same result but without the normal ordering and with $c(s)$ set to zero. *This proposition is a precise version of the Mayer expansion.* We will elaborate on this in the last section.

Proof. The action of convolution by μ on the initial data is

$$\mu_t * V^{(0)} = z \sum_x \exp\left[-\frac{1}{2} \int_0^t \dot{C}_{xx}(s)\, ds\right] :\exp(i\varphi_x):_T$$

so Eq. (2.15) in the proof of Theorem 2.2 can be replaced by

$$v(t) = \exp\left[-\frac{1}{2} \int_0^t c(s)\, ds\right] v(0) + \frac{1}{2} \int_0^t ds\, |\dot{C}(s)| \left[\frac{\partial v}{\partial \varphi}(s)\right]^2$$

Provided $t \leqslant T$, we can replace this with the majorant

$$v(t) = \exp\left[-\frac{1}{2}\int_0^t c(s)\,ds\right] v(0) + \int_0^t ds\,|\dot{C}(s)|\,k(s) \times \exp\left[-\frac{1}{2}\int_s^t c(\tau)\,d\tau\right]\left[\frac{\partial}{\partial\varphi}v(s)\right]^2$$

where

$$k(s) \equiv \exp\left[\frac{1}{2}\int_s^T c(\tau)\,d\tau\right]$$

This integral equation corresponds to

$$\frac{\partial v}{\partial t} = -\frac{1}{2}c(t)\,v + \frac{1}{2}k(t)\,|\dot{C}(t)|\left(\frac{\partial v}{\partial\varphi}\right)^2, \; v(0) = \sum_{M=1}^{\infty} V_M(0)\,\varphi^M$$

We make the change of variables

$$v(t) = \exp\left[\frac{1}{2}\int_t^T c(s)\,ds\right]\bar{v}(t)$$

to eliminate the linear term in this equation and proceed as we did with (2.16) in the proof of Theorem 2.2, to conclude that for $M \geqslant 1$

$$V_M(t) \leqslant \exp\left[\frac{1}{2}\int_t^T c(s)\,ds\right]\bar{v}_M(t)$$

where $\bar{v}(t)$ solves, for $t \leqslant T$,

$$\frac{\partial\bar{v}}{\partial t} = \frac{1}{2}\exp\left[\int_t^T c(s)\,ds\right]|\dot{C}(t)|\left(\frac{\partial\bar{v}}{\partial\varphi}\right)^2$$

$$\bar{v}(t=0) = \exp\left[-\frac{1}{2}\int_0^T c(s)\,ds\right]\sum_{M=1}^{\infty} V_M(0)\,\varphi^M = z\sum\frac{\varphi^M}{M!} = ze^{\varphi}$$

and by the action principle this equation has a unique solution at least for t such that

$$\int_0^t |\dot{C}(s)|\exp\left[\int_s^T c(\tau)\,d\tau\right]ds\,|z| < e^{-1}$$

Collecting these equations completes the proof. ∎

Constants. With initial data like $\exp(i\lambda\varphi)$ whose Nth derivative is bounded by const^N as opposed to $N!$, we can replace the 1/4 by $1/e$ because the initial data can be majorized by the series for e^{φ} instead of $-z\log(1-\varphi)$ and then the equations $v_t = \frac{1}{2}v_{\varphi}^2$, $v(0) = e^{\varphi}$ have a solution that exists for $t < 1/e$.

3. EXPLICIT FORMULAS AND THE MAYER EXPANSION

In this section we will obtain explicit formulas for the expansion that results from iterating the integral equation form of the flow equations of the last section. In the case of trigonometric initial data this expansion is the Mayer expansion and we concentrate on this aspect first.

For the special case discussed in extension 2.5, where the initial data $V^{(0)}$ are given by

$$V^{(0)} = \sum_{x \in \Lambda} :\exp(i\varphi_x):$$

there is the following well-known and easily proved identity, the *sine-Gordon transformation* (see, e.g., Ref. 10, Section 2), which connects the Gaussian integrals we have just been discussing to statistical mechanics. Let μ be a Gaussian measure with covariance u; then

$$\mu * \exp(-zV^{(0)}) = \sum_N \frac{z^N}{N!} \sum_{x_1,\dots,x_N} \exp\left[-\frac{1}{2}\sum_{i,j;i\neq j} u(x_i, x_j)\right] \exp\left(i\sum_j \varphi_{x_j}\right)$$

The right-hand side is the partition function for a grand canonical ensemble of N particles in states $x_1,\dots, x_N$ with activities $z\exp(i\varphi)$ and two-body interactions $u(x, y)$. The right-hand side of this equation can be defined in a wider context than the left, so we make the following generalizations.

Let $(\Lambda, d\rho(x))$ be an arbitrary finite measure space (the possible states of a single particle). Given (jointly) measurable real-valued functions $u(x, y)$ and $\varphi(x)$, we set

$$U \equiv U(x_1,\dots, x_N) \equiv \frac{1}{2}\sum_{i\neq j} u(x_i, x_j)$$

$$Z(\varphi) = \sum \frac{1}{N!}\int d^N\rho \exp(-U) \exp\left[i\sum \varphi(x_j)\right] \tag{3.1}$$

$$|u| \equiv \sup_x \int d\rho(y)\, |u(x, y)|$$

We assume that the interaction u is a family $u \equiv u(t)$ which is *stable* in the following sense: there exists $\dot{u}(\tau, x, y)$ and $c(\tau)$ such that for all N, $x_1, \dots, x_N$,

$$u(t, x, y) = \int_{-\infty}^{t} \dot{u}(\tau, x, y)\, d\tau$$
$$\dot{U}(\tau, x_1, \dots, x_N) \equiv \frac{1}{2} \sum_{i \neq j} \dot{u}(\tau, x_i, x_j) \geqslant -\frac{1}{2} c(\tau) N \tag{3.2}$$

It is a standard procedure, discussed, for example, in Ref. 7, to define the *Ursell coefficients* $(\exp[-U])_c$ by

$$(\exp[-U(x_1, \dots, x_N)])_c \equiv \sum_G \prod_{ij \in G} \{\exp[-u(x_i, x_j)] - 1\}$$
$$\equiv 1 \qquad \text{if} \quad N = 1$$

where G is summed over all connected graphs on N labeled vertices $\{1, \dots, N\}$ with bonds denoted by ij, where $i < j$, $i, j \in G$. Then one proves (Ref. 7, Appendix A), that *in the sense of formal power series*

$$\log Z(\varphi) = \sum \frac{1}{N!} \int d^N\rho\, (\exp[-U(x_1, \dots, x_N)])_c \exp\left[i \sum \varphi(x_j)\right]$$

This (Fourier) expansion is called the *Mayer expansion.*

Notation. Let $x_1, \dots, x_N$ be given. Given a subset $I \subset \{1, \dots, N\}$, we set

$$U(I) \equiv U((x_i)_{i \in I})$$
$$u_{ij} \equiv u(x_i, x_j)$$
$$ij \equiv \text{pair } (i, j) \text{ with } i < j$$

Our first result does not require the stability bound in (3.2), since it is an identity which holds pointwise in $x_1, \dots, x_N$.

Theorem 3.1. The Ursell coefficients are equal to

$$(\exp[-U(x_1, \dots, x_N)])_c$$
$$= (-1)^{N-1} \sum_T \prod_{b \in T} \int_{-\infty}^{t} ds_b\, \dot{u}_b(s_b) \exp\left[-\sum_{kl} \int_{s(k,l)}^{t} ds\, \dot{u}_{kl}(s)\right]$$

where T is summed over all connected tree graphs on the labeled vertices $\{1, \dots, N\}$ and b runs over all bonds in T. The $s(k, l)$ is defined by

$$s(k, l) \equiv \sup\{s_b : b \in \text{unique path in } T \text{ joining } k \text{ and } l\}$$

This identity is a better version of the tree graph identities found by Battle and Federbush[8] (see also Ref. 7, Theorem 3.1, and review in Ref. 7). An important feature of this identity is that if U is stable as in (3.2), then

$$\sum_{kl:\, s(k,l) \leqslant s} \dot{u}_{kl}(s) \geqslant -\frac{1}{2} c(s)\, N$$

In particular, this means that the exponent in Theorem 3.1 is bounded below, so that

$$\begin{aligned} &|(\exp[-U(x_1,..., x_N)])_c| \\ &\quad \leqslant \sum_T \prod_{b \in T} \int_{-\infty}^t ds_b\, |\dot{u}_b(s_b)| \exp\left[\frac{1}{2} \int_{-\infty}^t d\tau\, c(\tau)\, N\right] \end{aligned} \tag{3.3}$$

We postpone the proof of Theorem 3.1.

An Application of Theorem 3.1

The Villain Yukawa Gas. This model is related by Poisson summation to the massive Z ferromagnet. The analysis of this model was an essential step in the papers by Göpfert and Mack[12] on permanent confinement in Abelian $U(1)$ lattice gauge theory. The partition function of the model is

$$Z_\Omega(\varphi) = \sum_m \exp[i\beta^{1/2}(m, \varphi)] \exp[-\beta(m, vm)/2]$$

where m is summed over all assignments $x \to m(x) \in \mathbf{Z}$ of integers to sites in a finite subset Ω of a simple cubic lattice $\mathbf{Z}^\nu$ in ν dimensions and $v = (-\Delta + M^2)^{-1}$. We are using the notation $(m, \varphi) \equiv \sum_x m(x)\, \varphi(x)$. Define $\Lambda \equiv \Omega \times (\mathbf{Z} \setminus \{0\})$ and let $d\rho$ be counting measure on Λ. A typical point in Λ will be denoted (x, m) or ζ. It should be thought of as a charge m at site x.

Proposition 3.2 (Göpfert and Mack). For β sufficiently large, depending on dimension ν and the mass M,

$$\begin{aligned} \log Z_\Omega(\varphi) = \sum \frac{1}{N!} \int d^N\rho\, (\exp[-U(\zeta_1,..., \zeta_N)])_c \\ \times \exp\left[i\beta^{1/2} \sum m_j \varphi(x_j)\right] \end{aligned}$$

is convergent uniformly in Ω after dividing by the volume of Ω and

$$\begin{aligned} &\int_{x_1 \text{ fixed}} d^N\rho\, |(\exp[-U(\zeta_1,..., \zeta_N)])_c| \\ &\quad \leqslant (1 + \beta M^{-2})^{-1}\, C(\beta, M)^N\, (N-2)! \end{aligned}$$

where

$$C(\beta, M) = (1 + \beta M^{-2}) \sum_{m \geqslant 1} m \exp\left(-\frac{1}{2}\beta\varepsilon m^2 + m\right)$$

$$\varepsilon = (M^2 + 4\nu)^{-1}$$

Göpfert and Mack also give results on the exponential decay of the connected parts, which can easily be extracted from this approach also.

Proof. The essential point is that the self-energy of a charge forces the activity down. To exploit this it is important to use the hard core interaction that will shortly be introduced to prevent equal and opposite charges from sitting on top of each other.

Following Göpfert and Mack, we note that this is also the grand canonical partition function of a hard core Yukawa gas of particles on Ω with infinitely many species labeled by charges $m = \pm 1, \pm 2$, etc., so that the possible states are labeled by $\xi \equiv (x, m) \in \Lambda \equiv \Omega \times (\mathbf{Z} \backslash \{0\})$ and

$$Z_\Omega(\varphi) = \sum_N \frac{1}{N!} \int d^N\rho \prod_{i=1}^{N} \exp[i\beta^{1/2} m_i \varphi(x_i)] \exp(-U)$$

with $d\rho$ a counting measure on Λ and

$$U(\xi_1, ..., \xi_N) = \frac{1}{2} \sum_{i,j} u(\xi_i, \xi_j)$$

where

$$u(\xi_i, \xi_j) = \beta m_i m_j v(x_i, x_j) + v_{\mathrm{hc}}(x_i, x_j)$$

$$v_{\mathrm{hc}}(x_i, x_j) = \infty \qquad \text{if} \quad x_i = x_j$$

$$= 0 \qquad \text{otherwise.}$$

Note that, in contrast to (3.1), self-energies are included in U.

Since this is a lattice system, $v^{-1} = -\Delta + M^2$ is a bounded operator. We set $\varepsilon^{-1} = \|v^{-1}\| = M^2 + 4\nu$. This implies that as a form $v \geqslant \varepsilon$, so that $U \geqslant \frac{1}{2}\varepsilon\beta(\sigma, \sigma)$, where $\sigma(x) = \sum_i m_i \,\delta(x - x_i)$. Furthermore, because of the hard core, $(\sigma, \sigma) \geqslant \sum m_i^2$, so we have the stability estimate

$$U \geqslant \frac{1}{2} \varepsilon\beta \sum m_i^2$$

This lower bound by a positive quantity will drive the activity down.

We define for $t \in [-1, 1]$

$$\begin{aligned} \dot{u}(t, \xi_i, \xi_j) &= \dot{f}(t)\, \delta(x_i - x_j) \qquad \text{for} \quad 1 \geqslant t > 0 \\ &= \beta m_i m_j v(x_i, x_j) \qquad \text{for} \quad 0 \geqslant t \geqslant -1 \end{aligned}$$

where $f(t)$ is any monotone function with $f(1) = \infty$ and $f(0) = 0$, so that $u = \int \dot{u}\, dt$. By Theorem 3.1,

$$\begin{aligned} &|(\exp[-U(\xi_1, \ldots, \xi_N)])_c| \\ &\quad \leqslant \sum_T \left| \int_{-1}^{1} \prod_{b \in T} [ds_b\, \dot{u}_b(s_b)] \exp\left[-\frac{1}{2} \sum_{k,l} \int_{s(k,l)}^{1} ds\, \dot{u}_{kl}(s) \right] \right| \end{aligned}$$

where the sum over k, l now includes $k = l$ terms and $s(k, l) = -1$ if $k = l$.

For fixed (s_b) let n be the set of bonds kl $(k \neq l)$ for which $s(k, l) \leqslant 0$ and let p be the set of bonds for which $s(k, l) > 0$. Let $T_+ = T \cap p$, $T_- = T \cap n$. Rewrite the exponent using

$$\begin{aligned} &\sum_{k,l} \int_{s(k,l)}^{1} ds\, \dot{u}_{kl}(s) \\ &\quad = \sum_{k,l \in p} \int_{s(k,l)}^{1} ds\, \dot{u}_{kl}(s) \\ &\quad\quad + \sum_{k,l \in n} \int_{s(k,l)}^{0} ds\, \dot{u}_{kl}(s) + \sum_{k,l \in n} \int_{0}^{1} ds\, \dot{u}_{kl}(s) \end{aligned}$$

The last term integrates up to a hard core between all pairs k, l with $kl \in n$. The second term equals

$$\sum_{k,l} \int_{s(k,l)}^{0} ds\, \dot{u}_{kl}(s) = \int_{-1}^{0} ds \sum_{k,l} \dot{u}_{kl}(s)\, \chi(s \geqslant s(k, l))$$

where χ is the indicator function. It follows from the definition of $s(k, l)$ that for each s there exists a partition of $\{1, \ldots, N\}$ into disjoint subsets $X_1, \ldots, X_r$ such that

(a) k, l belong to the same X iff $s(k, l) \leqslant s$.

(b) $\sum_{k,l} \dot{u}_{kl}(s)\, \chi(s \geqslant s(k, l)) = \sum_{i=1}^{r} \sum_{k,l \in X_i} \dot{u}_{kl}(s)$.

Property (a) implies that if $i, j \in$ the same X, then $s(i, j) \leqslant s \leqslant 0$, so therefore $ij \in n$ and the last term forces a hard core between i and j. Therefore, by the stability estimate, the second term is bounded below by $\varepsilon\beta \sum m_i^2$. The first term is the sum of positive terms, so we can bound it below by dropping all terms except those with $k, l \in T_+$.

The last paragraph shows that

$$
\begin{aligned}
&|(\exp[-U(\xi_1,\ldots,\xi_N)])_c| \\
&\quad \leqslant \sum_T \int_{-1}^{1} \prod_{b\in T} ds_b \prod_{b\in T_+} \left\{ \dot{u}_b(s_b) \exp\left[-\int_{s_b}^{1} \dot{u}_b(s)\, ds \right] \right\} \\
&\quad \times \prod_{b\in T_-} [|\dot{u}_b(s_b)|] \exp\left(-\frac{\beta}{2} \varepsilon \sum m_i^2 \right)
\end{aligned}
$$

We hold T, T_+, T_- fixed and do the ds_b integrals, noting that if $b=ij$ and we set $x_b = x_i - x_j$, then

$$
\int_0^1 ds_b\, \dot{u}_b(s_b) \exp\left[-\int_{s_b}^{1} \dot{u}_b(s)\, ds \right] = \delta(x_b)
$$

Therefore

$$
\begin{aligned}
&|(\exp[-U(\xi_1,\ldots,\xi_N)])_c| \\
&\quad \leqslant \sum_T \sum_{T_+ \subset T} \prod_{b\in T_+} \delta(x_b) \prod_{ij\in T_-} m_i m_j \beta v(x_{ij}) \exp\left(-\frac{\beta}{2} \varepsilon \sum m_i^2 \right)
\end{aligned}
$$

Let $d(i, T) = \#$ of lines in T that meet vertex i. Then

$$
\prod_{ij\in T_-} m_i m_j \leqslant \prod_i |m_i|^{d(i,T)}
$$

and this combines with the last inequality to give us

$$
\begin{aligned}
&|(\exp[-U(\xi_1,\ldots,\xi_N)])_c| \\
&\quad \leqslant \sum_T \prod_{b\in T} [\delta(x_b) + \beta v(x_{ij})] \prod_i |m_i|^{d(i,T)} \exp\left(-\frac{\beta}{2} \varepsilon m_i^2 \right)
\end{aligned}
$$

We sum both sides of this inequality over $x_2,\ldots, x_N$, using

$$
\sum_{x_2,\ldots,x_N} \prod_{b\in T} [\delta(x_b) + \beta v(x_{ij})] \leqslant (1+\beta M^{-2})^{N-1}
$$

We estimate the sum over trees T using Cayley's theorem, which says that the number of trees on N vertices with $d(i, T) = d(i)$ is $(N-2)!/\prod(d(i)-1)!$. The result is

$$
\begin{aligned}
&\int_{x_1 \text{ fixed}} d^N\rho\, |(\exp[-U(\xi_1,\ldots,\xi_N)])_c| \\
&\quad \leqslant (1+\beta M^{-2})^{N-1} (N-2)! \left[\sum_{m, d\geqslant 1} \frac{|m|^d}{(d-1)!} \exp\left(-\frac{\beta}{2} \varepsilon m^2 \right) \right]^N \\
&\quad = (1+\beta M^{-2})^{-1}\, C(\beta, M)^N\, (N-2)! \quad \blacksquare
\end{aligned}
$$

Theorem 3.1 is proved using the following lemma, which involves the flow equations of Section 2 written in terms of the Fourier coefficients of V.

Lemma 3.3. Fix $x_1,..., x_N$. The system of ordinary differential equations for the functions $f(I) \equiv f(t, I)$,

$$\frac{d}{dt} f(I) = - \sum_{i,j \in I, i<j} \dot{u}_{ij} f(I) - \frac{1}{2} \sum_{J \subset I} \sum_{i \in J, j \in I \setminus J} \dot{u}_{ij} f(J) f(I \setminus J)$$

where $I \subset \{1,..., N\}$, together with the initial conditions

$$\lim_{t \to -\infty} f(t, I) = 1 \qquad \text{if} \quad |I| = 1$$
$$= 0 \qquad \text{otherwise}$$

has a unique solution, which is

$$f(t, I) = (e^{-U(t,I)})_c$$

The sum over $J \subset I$ extends over proper subsets of I.

Proof. We begin by showing that the Ursell function is a solution. For $|I| = 1$ the differential equation is $df/dt = 0$, which is satisfied by the $N = 1$ Ursell functions. For $N > 1$ we use the definition to compute that

$$\frac{d}{dt}(e^{-U(I)})_c = -\sum_G \sum_{b \in G} \dot{u}_b e^{-u_b} \prod_{\substack{a \in G \\ a \neq b}} (e^{-u_a} - 1)$$
$$= -\sum_b \dot{u}_b \sum_G \prod_{a \in G} (e^{-u_a} - 1)$$
$$+ \sum_b \dot{u}_b \left[\sum_{G: b \notin G} \prod_{a \in G} (e^{-u_a} - 1) - \sum_{G: b \in G} \prod_{\substack{a \in G \\ a \neq b}} (e^{-u_a} - 1) \right]$$

The first term is $-\sum_b \dot{u}_b (e^{-U(I)})_c$, which is the first term in the right-hand side of the differential equation. To each graph G in the first sum inside the square brackets we may associate a graph G' in the second sum by $G' \equiv G \cup \{b\}$. The corresponding terms cancel. The remaining terms in the second sum are labeled by graphs G such that $b \in G$ and G becomes disconnected when b is removed. Let G_1 and G_2 be the two connected components of G with b removed. Let J be the subset of vertices in I that are connected by G_1. Then we can write all the surviving terms in the square brackets as

$$-\sum_{kl} \dot{u}_{kl} \sum_{J: k \in J, l \in I \setminus J} \sum_{G_1 \text{ on } J} \sum_{G_2 \text{ on } I \setminus J} \prod_{a \in G_1 \cup G_2} (e^{-u_a} - 1)$$

Using the definition of Ursell functions, we find that this equals

$$-\sum_{kl} \dot{u}_{kl} \sum_{J:\, k \in J, l \in I \setminus J} (e^{-U(J)})_c \, (e^{-U(I \setminus J)})_c$$

which is the second term on the right-hand side in the differential equation. Thus, we have proved that the Ursell functions satisfy the differential equation.

To see that the differential equation has a unique solution, suppose that $g(I)$ is another solution with the same initial conditions. If $|I| = 1$, then certainly $g(I) = f(I) = 1$ for all t. Since the nonlinear term on the right-hand side of the equation involves $g(J)$ with $|J| < |I|$, we find

$$\frac{d}{dt}[f(I) - g(I)] = -\sum_{ij:\, i,j \in I} \dot{u}_{ij}[f(I) - g(I)]$$

and so $f = g$ for all t. ∎

Proof of Theorem 3.1. For $I \subset \{1,..., N\}$ let

$$f(I) \equiv (-1)^{|I|-1} \sum_T \prod_{b \in T} \int_{-\infty}^{t} ds_b \, \dot{u}_b(s_b) \exp\left[-\sum_{kl} \int_{s(k,l)}^{t} ds \, \dot{u}_{kl}(s)\right]$$

$$\equiv 1 \qquad \text{if} \quad |I| = 1$$

where T is summed over all connected tree graphs whose vertices are the elements of I. By Lemma 3.2 it suffices to prove that f satisfies the system of differential equations and the initial condition in the lemma. The initial conditions are satisfied trivially.

When we differentiate f with respect to t the derivative can either act on one of the ds_b integrals or on the exponential function. The result of the latter is $-\sum \dot{u}_{ij} f(I)$, which is the first term in the differential equation. The contribution from the derivative acting on the ds_b integrals is

$$(-1)^{|I|-1} \sum_T \sum_{b' \in T} \dot{u}_{b'}(t) \prod_{\substack{b \in T \\ b \neq b'}} \int_{-\infty}^{t} ds_b$$

$$\times \prod_{\substack{b \in T \\ b \neq b'}} \dot{u}_b(s_b) \exp\left[-\sum_{ij} \int_{t(i,j)}^{t} d\tau \, \dot{u}_{ij}(\tau)\right]$$

where $s_{b'} = t$. If we remove b' from T, then the set I splits into two subsets J and $I \setminus J$, which label the variables connected by the two connected subtrees into which $T \setminus \{b'\}$ splits. If $i \in J$ and $j \in I \setminus J$, then the path from i to j in T must contain b'. This implies that $t(i, j) = t$ and so $\int_{t(i,j)}^{t} ds \, \dot{u}_{ij} = 0$. Thus the

ds_b integrals factor into integrals associated with J and $I\backslash J$. At the same time we can factor the sum over T into a sum over T_1 and T_2, all tree graphs on J and $I\backslash J$, respectively. The sum over T_1 is $f(J)$ and the sum over T_2 is $f(I\backslash J)$, so we obtain

$$-\frac{1}{2}\sum_J \sum_{\substack{i\in J\\ j\in I\backslash J}} \dot{u}_{ij} f(J)\, f(I\backslash J)$$

This is the other term in the differential equation, so we have shown that f satisfies the differential equation and the proof is complete. ∎

Now we consider the analogous result in the context of perturbations of Gaussian measures. In particular we establish a relationship between the flow equation and the Hamilton–Jacobi equation (with the "right" sign) term by term in an infinite series. Given a tree graph T on vertices labelled $1,\dots, n$ and nonnegative numbers $(s_b)_{b\in T}$, we construct from $\boldsymbol{\varphi}$ a new Gaussian process $\boldsymbol{\varphi}\equiv(\boldsymbol{\varphi}_i)\equiv(\varphi_{x,i})_{x\in \Lambda, i=1,\dots,n}$ with joint Gaussian distribution $\mu_{T,s}$ defined by the covariance

$$C_{x,i,y,j}\equiv\int_{s(i,j)}^{t} \dot{C}_{xy}(s)\, ds$$

Let $A_1,\dots, A_n$ be given functions of $\boldsymbol{\varphi}$; then we define a new function of $\boldsymbol{\varphi}$, $(A_1,\dots, A_n)_c$, which is multilinear in the A's, by

$$(A_1,\dots, A_n)_c\equiv\sum_T \prod_{b\in T}\left(\int_0^t ds_b\right)\mu_{T,s} * \left\{\prod_{b\in T}[\dot{\Delta}_b(s_b)]\prod_{i=1}^{n} A_i\right\}$$

where on the right-hand side $A_i\equiv A_i(\boldsymbol{\varphi}_i)$ and $\mu_{T,s}\,*$ is convolution by the measure defined above in the following sense:

$$(\mu_{T,s} * F)(\boldsymbol{\varphi})=\int d\mu_{T,s}\, F(\boldsymbol{\varphi}+\boldsymbol{\varphi}_1,\dots, \boldsymbol{\varphi}+\boldsymbol{\varphi}_n)$$

and if the bond $b\in T$ is ij, then

$$\dot{\Delta}_b\equiv\sum_{x,y} \dot{C}_{xy}\, \partial/\partial\varphi_{x,i}\, \partial/\partial\varphi_{y,j}$$

Theorem 3.4. If $V=V(t, \boldsymbol{\varphi})$ solves

$$\frac{\partial V}{\partial t}=\frac{1}{2}\sum \dot{C}_{xy}(t)\left(\frac{\partial^2 V}{\partial\varphi_x\,\partial\varphi_y}-\frac{\partial V}{\partial\varphi_x}\frac{\partial V}{\partial\varphi_y}\right)$$

with given initial data, then, as a formal power series,

$$V(t) = -\sum \frac{1}{n!} (V(t=0), \ldots, V(t=0))_c$$

The series is convergent for initial data and t as in Theorem 3.1.

If the convolution by μ were omitted in the definition of $(V(0), \ldots, V(0))_c$, then the series would solve the Hamilton–Jacobi equation instead, i.e., the flow equation without the second-order derivatives in φ. This will be clear from the proof and it is a way of seeing that the solution to the Hamilton–Jacobi equation (which is also the classical action) is the sum of all tree graphs.

Proof. Let f_n denote the nth coefficient of the series, i.e.,

$$f_n = \frac{1}{n!} \sum_T \prod_{b \in T} \left(\int_0^t ds_b \right) \mu_{T,s} * \left\{ \prod_{b \in T} [\dot{\Delta}_b(s_b)] \prod_i A_i \right\}$$

where $A_i = -V(t=0, \boldsymbol{\varphi}^{(i)})$. We substitute the series into the equation and equate powers of V to find that the differential equation is satisfied if

$$\dot{f}_n = \frac{1}{2} \sum \dot{C}_{xy} \left[\frac{\partial^2}{\partial \varphi_x \, \partial \varphi_y} f_n - \sum_{l,m:\, l+m=n} \left(\frac{\partial}{\partial \varphi_x} f_l \right) \left(\frac{\partial}{\partial \varphi_y} f_m \right) \right]$$

$\dot{f}_n$ has two terms, depending on whether the time derivative acts on the t in the $\mu_{T,s}$ or the t in the limits of the ds integrals. The first term is simplified using

$$\frac{d}{dt} \mu_{T,s} * F = \frac{1}{2} \dot{\Delta} \mu_{T,s} * F$$

and is the first term on the left-hand side of the differential equation. When the t derivative acts on the limit of a particular ds_b integral we lose the integral and set $s_b = t$. If we remove the bond b from the tree graph T, then the set of vertices $\{1, \ldots, n\}$ splits into two sets J and J^c each of which is connected by the (two) subtrees into which T dissociates. If i belongs to J and j belongs to J^c, then the path in T from i to j must contain b, so that $s(i, j) = t$. This remark and the definition of the covariance of $\mu_{T,s}$ imply that the random variables $\boldsymbol{\varphi}^{(i)}$ with i in J are independent of the variables with i in J^c. If the bond b and the subset J are held fixed, then the sum over trees T factors into two independent sums over trees on J and J^c. Since $\mu_{T,s}$ also factors, we can write the sum over all terms with $b = ij$ fixed as

$$\sum_{J:\, i \in J,\, j \in J^c} \frac{l!\, m!}{n!} \sum_{x,y} \dot{C}_{x,y}\, g(i, x, J)\, g(j, y, J^c)$$

where $l = |J|$ and $m = |J^c|$ and

$$g(i, x, J) = \frac{1}{l!} \sum_{T \text{ on } J} \prod_{a \in T} \left(\int_0^t ds_a \right) \mu_{T,s} * \frac{\partial}{\partial \varphi_{x,i}} \left\{ \prod_{a \in T} [\dot{\Delta}_a(s_a)] \prod_{k \in J} A_k \right\}$$

We now sum over the bond ij, interchange the sums over ij and J, and use

$$\sum_{i \in J} g(i, x, J) = \frac{\partial}{\partial \varphi_x} f_l$$

$$\sum_{J: |J| = l} \frac{l! \, m!}{n!} = 1$$

The result is the second term in the differential equation. ∎

4. THE YUKAWA GAS

It has been shown[6,10,13–17] that the two-dimensional Yukawa gas has an infinite series of thresholds in the interval $\beta \in [4\pi, 8\pi]$, starting at $\beta = 4\pi$, at each of which, successive terms in the Mayer expansion, beginning with the term of order z^2, become infinite. However, the corresponding sine-Gordon quantum field theory remains stable provided an extra vacuum energy counterterm is included. They have interpreted these thresholds in terms of a sequence of collapses of the system into dipoles, quadrupoles, etc., and Benfatto *et al.* have started a program to study the convergence properties of the Mayer expansion for the Yukawa gas in the region $4\pi < \beta < 8\pi$. In this section we will show by an extension of our previous arguments that the Mayer expansion is convergent for β in the range $4\pi \leqslant \beta < 16\pi/3$ provided the $O(z^2)$ term is omitted and we will make some remarks on how to extend the argument up to $\beta < 6\pi$, which is the first threshold after 4π. Benfatto *et al.*[16] have obtained similar results independently. First it is helpful to see how we can obtain their result on the convergence of the Mayer expansion for $\beta < 4\pi$. We begin by proving a general theorem of independent interest, which appeared in Ref. 9 in a slightly weaker form.

We assume we have a system that is stable in the sense of (3.2) and we define $R_s = \{\tau \in \mathbf{R}: t \geqslant \tau \geqslant s$ and $\dot{v}$ is a purely repulsive interaction$\}$ and then set

$$\hat{u}(s) = \dot{u}(s) \exp\left(-\int_{R_s} \dot{u} \right)$$

We also note that if $\tau \in R_s$, then $c(\tau) = 0$.

Theorem 4.1. Let

$$Q \equiv \int_{-\infty}^{t} |\hat{u}(s)| \exp\left[\int_{s}^{t} c(\tau)\, d\tau\right] ds$$

Then

$$\sup_{x} \int d^{N-1}\rho \,|(\exp[-U(t, x, x_1,..., x_{N-1})])_c| \leqslant N^{N-2} Q^{N-1}$$

If $Q < e^{-1}$, then the Mayer expansion converges.

The appearance of $\hat{u}$ as opposed to $\dot{u}$ in the norm is quite valuable, since the theorem can be applied to systems with hard cores, unlike the theorem in Ref. 9. Apart from this, this result was in essence already obtained in Section 2 (Extension 2.5), except that we did not formulate a continuum version and we imposed the slightly stronger condition that $\dot{u} \equiv \dot{C}$ be positive definite as opposed to stable. (The imposition of initial data at $t = -\infty$ instead of $t = 0$ is a trivial change.) Theorem 3.2 is also easily derived from Theorem 4.1.

Proof of Theorem 4.1. We return to the flow equation of Lemma 3.3 and write it as an integral equation:

$$f(I, t) = -\frac{1}{2} \int_{-\infty}^{t} ds \sum_{J \subset I} \sum_{\substack{i \notin J \\ j \in J}} \dot{u}_{ij}(s)\, f(J, s)\, f(I \backslash J, s) \exp\left(-\sum_{kl} \int_{s}^{t} \dot{u}_{kl}\right)$$

provided $|I| > 1$; if $|I| = 1$, then $f(I, t) = 1$; if $I = \emptyset$, then $f(I, t) = 0$. To prove this, it suffices to check that it satisfies the differential flow equation.

We introduce the norms

$$F_n(s) \equiv \sup_{x_1} \int dx_2 \cdots dx_n\, |f(I, s)|$$

(with the integral and sum omitted if $n = 1$) and make the *inductive assumption*: for $2 \leqslant k < n$,

$$F_k(s) \leqslant \left\{\int_{-\infty}^{s} ds'\, |\hat{u}(s')| \exp\left[\int_{s'}^{s} c(\tau)\, d\tau\right]\right\}^{k-1} k^{k-2}$$

We substitute this into the integrated flow equation and apply the stability estimates, $\sum_{kl} \dot{u}_{kl}(\tau) \geqslant -\frac{1}{2} c(\tau)\, n$ if τ not in R_s, $\geqslant \dot{u}_{ij}$ for any ij if τ is in R_s, to obtain

$$F_n(t) \leqslant \frac{1}{2} \int_{-\infty}^{t} ds \sum_k \binom{n}{k} k^{k-1} (n-k)^{n-k-1} |\hat{u}(s)|$$
$$\times \left\{ \int_{-\infty}^{s} ds' \, |\hat{u}(s')| \exp\left[\int_{s'}^{s} c(\tau)\, d\tau \right] \right\}^{n-2}$$
$$\times \exp\left[\frac{1}{2} n \int_{s}^{t} c(\tau)\, d\tau \right]$$

We now combine the exponents after increasing the second one by changing its coefficient from $n/2$ to $n-1$ and use the following combinatoric lemma:

Lemma 4.2. $\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1} (n-k)^{n-k-1} = (n-1)\, n^{n-2}$.

(We prove this result at the end of this section.) We find

$$F_n(t) \leqslant (n-1)\, n^{n-2} \int_{-\infty}^{t} ds \, |\hat{u}(s)| \left\{ \int_{-\infty}^{s} ds' \, |\hat{u}(s')| \right.$$
$$\left. \times \exp\left[\int_{s'}^{t} c(\tau)\, d\tau \right] \right\}^{n-2} \exp\left[\int_{s}^{t} c(\tau)\, d\tau \right]$$
$$= n^{n-2} \left\{ \int_{-\infty}^{t} ds' \, |\hat{u}(s')| \exp\left[\int_{s'}^{t} c(\tau)\, d\tau \right] \right\}^{n-1}$$

because the ds integral can be performed explicitly! This completes the inductive step.

Since $F_n(t)$ is the size of the nth Ursell coefficient by virtue of Lemma 3.3, the induction proves that

$$\sup_x \int d^{N-1}\rho \, |(\exp[-U(t, x, x_1, \ldots, x_{N-1})])_c| \leqslant Q^{N-1} N^{N-2}$$

which is the bound claimed in the theorem. The convergence of the Mayer expansion for $Q < e^{-1}$ is an immediate consequence of Sterling's theorem, and this ends the proof of Theorem 4.1. ∎

The Yukawa Gas for $\beta < 4\pi$ (The Cosine Euclidean Quantum Field Theory). We take the state space $(\Lambda, d\rho)$ to be $\Omega \times \{-1, 1\}$, where Ω is an open subset of $\mathbf{R}^2$. A point $\zeta \equiv (x, \varepsilon) \in \Lambda$ describes a charge with position $x \in \Omega$ and charge $\varepsilon = \pm 1$. The measure $d\rho$ is given by $\int d\rho = z \sum_{\varepsilon = \pm 1} \int_\Omega dx$. The interaction is

$$u(\zeta_1, \zeta_2) = \beta \varepsilon_1 \varepsilon_2 (1 - \Delta)^{-1} (x_1, x_2)$$

where

$$(1-\Delta)^{-1}(x, y)=\frac{1}{(2\pi)^2}\int d^2k \frac{1}{(k^2+1)} e^{ik\cdot(x-y)}$$

Using

$$\frac{1}{k^2+1}=\int_{-\infty}^{0} dt \frac{d}{dt} \frac{1}{k^2+e^{-t}}$$

to provide the representation $u=\int dt\, \dot{u}(t)$, we have stability in the sense of (3.2) *even though this model is not classically stable.* It is now routine to compute Q using the relations

$$|\dot{u}(s)|=2\beta e^{-s}; \qquad c(s)=\dot{u}(\zeta, \zeta)=\beta/4\pi$$

to show that the Mayer expansion converges provided

$$\beta<4\pi \qquad \text{and} \qquad 2|z|\, \beta(1-\beta/4\pi)^{-1}<e^{-1}$$

Apart from the explicit condition on z, this result was first obtained in Ref. 6. A direct proof that the Mayer expansion converges for this system was first obtained in Ref. 11.

By the sine-Gordon transformation, the partition function for this system is

$$Z(\varphi)=\lim_{T\to -\infty} \int d\mu_T(\varphi') \exp\left[z\int_\Omega dx :\cos \beta^{1/2}(\varphi+\varphi'):_T\right]$$

where the covariance of $d\mu_T$ is

$$\int_T^0 \frac{d}{dt}\frac{1}{k^2+e^{-t}}\, dt$$

and the normal ordering is with respect to this covariance. Thus, the convergence of the Mayer expansion uniformly as $T\to -\infty$ and as Ω increases to $\mathbf{R}^2$ together with the fact that each term in the expansion is Euclidean-invariant in these limits provides an easy proof of the Osterwalder–Schrader axioms (except for physical positivity) for this field theory. Physical positivity can also be obtained this way, but it is necessary to use a different type of short-distance cutoff, such as the lattice approximation, which has the positivity property, and prove that in the no-cutoff limit the two expectations are the same. Since each term in the Mayer expansion is a sum over a finite number of Mayer graphs, for each of which we have an explicit expression, we can show that graph by graph and therefore term by term in the Mayer expansion the limits are identical and then the same is true for $\log Z$ by our uniform convergence estimates on the Mayer expansion.

The Yukawa Gas for $\beta < 16\pi/3$. In a sense, the argument we have just given is based on the possibility of calculating the coefficient of the monopole $O(z)$ term in the expansion exactly (it equals one). Now we are about to follow the same outline, but the dipoles will be singled out for special consideration as well. This type of idea is basic to the program of Benfatto *et al.* and was also proposed in Ref. 12. It is possibly useful in any system where forces can cancel because there are both attractive and repulsive forces, but we will immediately specialize to the Yukawa system in two dimensions. Define ε by

$$c(s) = \beta/4\pi \equiv \tfrac{4}{3}(1-\varepsilon)$$

where $\varepsilon > 0$ since $\beta < 16\pi/3$. We make the *inductive assumption*: for $1 \leqslant k < n$, $k \neq 2$,

$$F_k(s) \leqslant C^{k-1} k^{k-2} e^{s(k-1)}$$

where $C \equiv C(\beta)$ is a constant depending only on β, which is determined below. This bound holds when $k = 1$ because $F_1 = 1$. When the induction is complete and the constant $C(\beta)$ is substituted in, we will have proved the following theorem:

Theorem 4.3. For $4\pi < \beta < 16\pi/3$ and $|z|\, eC(\beta) < 1$, where

$$C(\beta) \equiv \max\left(\frac{24\pi\beta}{16\pi - 3\beta}, \frac{2\beta}{3 - \beta/2\pi}\left(\int dx\, |\partial_x \dot{v}|\right)\left(\int dx\, |x|\, |\dot{v}|\right)\right)$$

$$\dot{v}(x) \equiv (1-\Delta)^{-2}(x) \equiv (2\pi)^{-2} \int d^2k\, (k^2+1)^{-2} e^{ik\cdot x}$$

the Mayer expansion for the two-dimensional Yukawa gas with $n = 2$ term omitted,

$$\sum_{n \neq 2} \frac{1}{n!} \int d^{n-1}\rho\, (e^{-U_n})_c\, (\zeta_1, ..., \zeta_n)$$

is absolutely convergent uniformly in the volume Ω. The $d\rho$ includes z and Ω, and is defined above.

Inductive Step. We consider the right-hand side of the integrated flow equation and divide the estimates into three cases:

1. $|J| \neq 2$ and $|I \backslash J| \neq 2$.
2. ($|J| = 2$ and the charges in J have the same sign) and/or ($|I \backslash J| = 2$ and the charges in $I \backslash J$ have the same sign).
3. ($|J| = 2$ and the charges in J have opposite sign) and/or ($|I \backslash J| = 2$ and the charges in $I \backslash J$ have opposite sign).

Cases 2 and 3 overlap; we assign the overlap to case 2. We write

$$F_n \equiv F_n^{(1)} + F_n^{(2)} + F_n^{(3)}$$

corresponding to the decomposition into these three cases.

Case 1. We substitute into the integrated flow equation the inductive assumption and apply the stability estimate just as was done above, to obtain

$$\begin{aligned} F_n^{(1)}(t) &\leqslant \frac{1}{2} \int_{-\infty}^{t} ds \sum_k \binom{n}{k} k^{k-1} (n-k)^{n-k-1} \\ &\quad \times 2\beta e^s C^{n-2} e^{s(n-2)} e^{2n(1-\varepsilon)(t-s)/3} \\ &= \beta C^{n-2} \sum \binom{n}{k} k^{k-1} (n-k)^{n-k-1} e^{(n-1)t} \\ &\quad \times \int_{-\infty}^{t} ds\, e^{-(t-s)(2\varepsilon n + n - 3)/3} \end{aligned}$$

Since $n \geqslant 3$, this is less than

$$\begin{aligned} &\beta C^{n-2} \sum \binom{n}{k} k^{k-1} (n-k)^{n-k-1} e^{(n-1)t} \int_{-\infty}^{t} ds\, e^{-(t-s)2\varepsilon n/3} \\ &\quad = \frac{1}{n} C^{n-1} \sum \binom{n}{k} k^{k-1} (n-k)^{n-k-1} e^{(n-1)t} \end{aligned}$$

provided $C \geqslant 3\beta/2\varepsilon$. The k was the cardinality of J and consequently omits the values 2 and $n-2$ in this case.

Case 2. Let $J = (k, l)$, so that the coordinates of the two particles in J are $x_k, \varepsilon_k, x_l, \varepsilon_l$ and in this case $\varepsilon_k = \varepsilon_l = \varepsilon$. Since $|J| = 2$, we can write $f(J)$ explicitly (by setting $|I| = 2$ in the integral flow equation):

$$f(J, s) = \int_{-\infty}^{s} ds'\, \dot{v}_{kl}(s')\, \varepsilon_k \varepsilon_l \exp\left[-\int_{s'}^{s} \dot{v}_{kl}(\tau)\, \varepsilon_k \varepsilon_l\, d\tau \right]$$

where

$$\dot{v}_{kl}(s) \equiv \dot{v}(s, x_k - x_l)$$

with

$$\dot{v}(s, x) \equiv e^{-s}(e^{-s} - \Delta)^{-2}(x) \equiv \dot{v}(e^{-s}x)$$

From this we see that

$$\sup_{x_k, \varepsilon} \int dx_1 \; |f(J, s)| \leqslant \beta e^s \leqslant \tfrac{1}{2} C e^s$$

provided $C \geqslant 2\beta$. In this case there is no exponential growth in s from a stability bound because both charges are the same. We find by substituting the stability bound and the inductive assumption and this bound into the right-hand side of the integrated flow equation that

$$F_n^{(2)}(t) \leqslant \frac{1}{2n} C^{n-1} \sum_{k=2,n-2} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} e^{(n-1)t}$$

Case 3. This is the nontrivial case. Let $J = (k, l)$ and the coordinates of the two particles are x_k, ε_k and x_l, ε_l with $\varepsilon_k = -\varepsilon_l = \varepsilon$. The stability bound together with the explicit formula for $f(J, s)$ given in case 2 imply

$$\begin{aligned} &\left| \sum_{j \in J} \dot{u}_{ij}(s) f(J, s) \right| \\ &\qquad \leqslant \left| \sum_{j=k,l} \dot{v}_{ij}(s)\, \varepsilon_j \right| \int_{-\infty}^{s} ds' \; |\dot{v}_{kl}(s')| \; e^{4(1-\varepsilon)(s-s')/3} \end{aligned}$$

We take advantage of J being a dipole by noting that

$$\begin{aligned} \left| \sum_{j=k,l} \dot{v}_{ij}(s)\, \varepsilon_j \right| &= |\dot{v}_{ik}(s) - \dot{v}_{il}(s)| \\ &\leqslant \int_{x_k}^{x_l} dx \; |(\partial_x \dot{v})(x - x_i, s)| \end{aligned}$$

Therefore

$$\begin{aligned} &\int dx_k\, dx_l \left| \sum_{j \in J} \dot{u}_{ij}(s) f(J, s) \right| \\ &\qquad \leqslant \int_{-\infty}^{s} ds' \; e^{4(1-\varepsilon)(s-s')/3} \int dx_k\, dx_l \int_{x_k}^{x_1} dx \\ &\qquad\quad \times |(\partial_x \dot{v})(x - x_i, s)| \; |\dot{v}_{kl}(s')| \end{aligned}$$

On the right-hand side one can see the following mechanism at work: either the dipole is "stretched" so that the $\dot{v}_{kl}$ factor is small, or $x_k - x_l$ is small, in which case the range of the dx integral is small and contributes a small factor. We can evaluate the integrals by

$$\int dx_k\, dx_l \int_{x_k}^{x_l} dx\, |(\partial_x \dot{v})(x - x_i, s)|\, |\dot{v}_{kl}(s')|$$

$$= \int dx_k\, dx_l \int_0^{x_l - x_k} dx\, |(\partial_x \dot{v})(x_k + x - x_i, s)|\, |\dot{v}_{kl}(s')|$$

By change of variables and translation invariance we can integrate over $x_k - x_i$, keeping x and $x_l - x_k$ fixed, followed by integration over x and then $x_l - x_k$:

$$= \int dx_{ki}\, |(\partial_x \dot{v})(x_{ki}, s)| \int dx_{lk}\, |x_l - x_k|\, |\dot{v}_{kl}(s')|$$

$$= e^{s/2 + 3s'/2} \int dx\, |(\partial_x \dot{v})(x, 0)| \int dx\, |x|\, |\dot{v}(x, 0)|$$

where we have used scaling. Therefore we have

$$\int dx_k\, dx_l \left| \sum_{j \in J} \dot{u}_{ij}(s)\, f(J, s) \right|$$

$$\leqslant \int dx\, |(\partial_x \dot{v})(x)| \int dx\, |x|\, |\dot{v}(x)|$$

$$\times \int_{-\infty}^{s} ds'\, e^{s/2 + 3s'/2} e^{4(1-\varepsilon)(s - s')/3}$$

$$\leqslant \frac{1}{2}\, C(2\beta e^s)\, e^s$$

provided

$$C \geqslant \frac{6}{\beta(1 + 8\varepsilon)} \int dx\, |(\partial_x \dot{v})(x)| \int dx\, |x|\, |\dot{v}(x)|$$

Note that the integral over s' would converge if $\varepsilon > -1/8$, which corresponds to $\beta < 6\pi$. We substitute this bound into the right-hand side of the integrated flow equation and combine it with the inductive assumption and the stability estimate to get

$$F_n^{(3)}(t) \leqslant \frac{1}{2n}\, C^{n-1} \sum_{k=2,n-2} \binom{n}{k} k^{k-1} (n-k)^{n-k-1}\, e^{(n-1)t}$$

We have completed the bounds in each case and by uniting them obtain

$$F_n(t) \leqslant \frac{1}{n}\, C^{n-1} \sum_{k=1}^{n} \binom{n}{k} k^{k-1} (n-k)^{n-k-1}\, e^{(n-1)t}$$

which by Lemma 4.1 is less than

$$\leqslant C^{n-1} n^{n-2} e^{(n-1)t}$$

which completes the inductive step, and ends the proof of Theorem 4.3. ∎

Remarks on $\beta < 6\pi$. As noted above, our $16\pi/3$ argument handles the $|J| = 2$ terms. However, if $16\pi/3 \leqslant \beta < 6\pi$, then the bounds we used in case 1 are divergent for $|J| = 3$ terms. These terms are "tripoles" and cannot be neutral, and in the left-hand side of the stability bound

$$\sum_{\substack{i<j \\ i,j \in J}} \dot{u}_{ij}(s) \geqslant -3\beta/8\pi$$

it should be possible to prove that the coefficient can be changed from a 3 to a 2, so that these terms are after all not divergent.

Beyond 6π. For $\beta > 6\pi$ the $|J| = 2$ terms are again divergent and the most obvious way to deal with them requires developing graphs with higher connectivity than trees in order to exhibit further cancellations. Now we run into difficulties with having too many graphs to estimate. This is the typical way in which the "large-field problem" of Gawedski and Kupiainen makes itself manifest and probably indicates that these methods are insufficient as they stand.

The Yukawa Gas in Three Dimensions. Imbrie[18] applied the iterated Mayer expansion of Göpfert and Mack to the Yukawa gas in three dimensions in order to prove screening for Coulomb systems in an extended region of the parameter space. This type of result on the Yukawa gas also follows from Theorem 4.1 with a deformation of the potential (like the one used in our discussion of the Villain Yukawa gas) as follows:

$$\begin{aligned} \dot{u}(s, \zeta_i, \zeta_j) &= \dot{f}(s)\, \chi(x_i - x_j) \qquad &&\text{for} \quad 1 \geqslant s > 0 \\ &= \beta \varepsilon_i \varepsilon_j \dot{v}(s, x_i, x_j) \qquad &&\text{for} \quad 0 \geqslant s \geqslant -\infty \end{aligned}$$

where χ is the characteristic function of the set $|x| \leqslant 1$, $f(s)$ is any monotone function with $f(1) = \infty$ and $f(0) = 0$, and

$$\dot{v}(s, x_1, x_2) = \frac{d}{ds} \frac{1}{-\Delta + L^{-2} e^{-s}} (x_1, x_2)$$

Note that $\int_{-\infty}^{1} d\tau\, \dot{u}$ is a Yukawa interaction of range L with a hard core of range l. We obtain stability estimates for this type of interaction by replacing the point charges by equivalent charge distributions on the surface of the hard cores using the electrostatic type of property that holds for the Yukawa interaction. The interaction energy is then bounded below by the sum of the self-energies of the spheres because the Yukawa interaction is positive definite. We assume $L \geqslant \beta \geqslant 1$, bound Q by a modest calculation, and find convergence of the Mayer expansion in a region of the form

$$c_1 z l^3 + c_2 z\beta^3 \exp(c_3\beta/l) + c_4 z\beta L^2 < 1$$

The standard methods can only obtain convergence in a region of the form

$$c_1 z l^3 + c_2 z\beta L^2 \exp(c_3\beta/l) < 1$$

which is much worse if $l \ll \beta \ll L$.

Proof of Lemma 4.2. Let $J \subset \{1,..., n\}$; then

$$\frac{1}{2}\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = \frac{1}{2}\sum_{J} \sum_{\substack{j \in J \\ i \notin J}} |J|^{|J|-2}\, |J^c|^{|J^c|-2}$$

We rewrite the left-hand side using Cayley's theorem, which says that the number of tree graphs T_1 on a set of vertices J is $|J|^{|J|-2}$:

$$= \frac{1}{2}\sum_{J} \sum_{\substack{j \in J \\ i \notin J}} \sum_{T_1 \text{ on } J} \sum_{T_2 \text{ on } J^c} 1$$

$$= (n-1)(\#\text{ of tree graphs on } \{1,..., n\})$$

because there is a two-to-one map from J, i, j, T_1, T_2 onto T, b, where T is a tree on $\{1,..., n\}$ and b is any bond in T. The map is given by $b = ij$ and $T \equiv T_1 \cup T_2$ and its inverse is to remove the bond b from T, thereby splitting it into two subtrees on J and J^c, respectively. The factor of two arises because of the ambiguity in how to name the two subsets of vertices J and J^c. ∎

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